

# Damped jump-telegraph processes

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## Abstract

We study a one-dimensional Markov modulated random walk with jumps. It is assumed that amplitudes of jumps as well as a chosen velocity regime are random and depend on a time spent by the process at a previous state of the underlying Markov process.

Equations for the distribution and equations for its moments are derived. We characterise the martingale distributions in terms of observable proportions between jump and velocity regimes.

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## 1. Introduction

Telegraph processes with different switchings and velocity regimes are studied recently in connection with possibility of different applications such as, for instance, queuing theory (see Zacks (2004), Stadje and Zacks (2004)) and mathematical biology (see Haderer (1999)). Special attention is devoted to financial applications (see Ratanov (2007), López and Ratanov (2012)). In the latter case, an arbitrage reasoning demands the presence of jumps.

The motions with deterministic jumps are studied in detail, see the formal expressions of the transition densities in Ratanov (2007), Di Crescenzo and Martinucci (2013). Such a model is developed for the option pricing problem, which is based on the risk-neutral approach, see Ratanov (2007). If the jump amplitudes are random, the case is less known. The telegraph processes of this type are studied earlier only under the assumption of mutual independence of jump values and jump amplitudes, see Stadje and Zacks (2004) and Di Crescenzo and Martinucci (2013). Similar setting were used for the purposes of financial applications, López and Ratanov (2012).

We present here a jump-telegraph process when an amplitude of the next jump depends on the (random) time spent by the process at the previous state. This approach is of special interest for the economical and the financial applications, everywhere when the comportment of process relates with friction and memory.

Assume that the particle moves with random (and *variable*) velocities performing jumps of random amplitude whenever the velocity is changed. More precisely, the actual velocity regime and the amplitude of the next jump are defined as (alternated) functions of the time spent by the particle at the previous state. We assume also that the time intervals between

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the subsequent state changes have sufficiently arbitrary alternated distributions. It creates an effect of damping process where a friction is generated by means of memory.

This setting generalises processes which were used before for market modelling by Ratanov (2007) and López and Ratanov (2012).

The underlying processes are described in Sections 2-3. Section 4 presents the result which can be interpreted as a Doob-Meyer decomposition. Several examples with different regimes of velocities and of jumps are presented.

## 2. Generalised jump-telegraph processes: distribution

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Consider two continuous-time Markov processes  $\varepsilon_0(t), \varepsilon_1(t) \in \{0, 1\}$ ,  $t \in (-\infty, \infty)$ . The subscript  $i \in \{0, 1\}$  indicates the initial state,  $\varepsilon_i(0) = i$  (with probability 1). Assume that  $\varepsilon_i = \varepsilon_i(t)$ ,  $t \in (-\infty, \infty)$  are left-continuous a. s.

Let  $\{\tau_n\}_{n \in \mathbb{Z}}$  be a Markov flow of switching times. The increments  $T_n := \tau_n - \tau_{n-1}$ ,  $n \in \mathbb{Z}$  are independent and possess alternated distributions (with the distribution functions  $F_0, F_1$ , the survival functions  $\bar{F}_0, \bar{F}_1$  and the densities  $f_0, f_1$ ). We assume that  $\tau_0 = 0$ , i. e. the state process  $\varepsilon_i$  is started at the switching instant. The distributions of  $\tau_n$  and  $T_n$  depend on the initial state  $i$ ,  $i \in \{0, 1\}$ . For brevity, we will not always indicate this dependence.

Consider a particle moving on  $\mathbb{R}$  with two alternated velocity regimes  $c_0$  and  $c_1$ . These velocities are described by two continuous functions  $c_i = c_i(T, t)$ ,  $T, t > 0$ ,  $i = 0, 1$ . At each instant  $\tau_n$  the particle takes the velocity regime  $c_{\varepsilon_i(\tau_n)}(T_n, \cdot)$ , where  $T_n$  is the (random) time spent by the particle at the previous state. We define a pair of the (generalised) telegraph processes  $\mathcal{T}_i$ ,  $i = 0, 1$  driven by variable velocities  $c_0, c_1$  as follows,

$$\begin{aligned} \mathcal{T}_0(t) &= \mathcal{T}_0(t; c_0, c_1) = \sum_{n=0}^{\infty} c_{\varepsilon_0(\tau_n)}(T_n, t - \tau_n) \mathbf{1}_{\{\tau_n < t \leq \tau_{n+1}\}}, \\ \mathcal{T}_1(t) &= \mathcal{T}_1(t; c_0, c_1) = \sum_{n=0}^{\infty} c_{\varepsilon_1(\tau_n)}(T_n, t - \tau_n) \mathbf{1}_{\{\tau_n < t \leq \tau_{n+1}\}}, \end{aligned} \quad t \geq 0. \quad (2.1)$$

The integral  $\int_0^t \mathcal{T}_i(s) ds$ ,  $i = 0, 1$  is named the *integrated telegraph process*.

Let  $N = N_i(t) := \max\{n \geq 0 : \tau_n \leq t\}$ ,  $t \geq 0$  be a counting process. Notice that,  $N_i(0) = 0$  and  $\varepsilon_0(t) = (1 - (-1)^{N_0(t)})/2$  and  $\varepsilon_1(t) = (1 + (-1)^{N_1(t)})/2$ .

The integrated telegraph process can be interpreted as the sum of random number of random variables. If  $N_i(t) = 0$ , then

$$\int_0^t \mathcal{T}_i(s) ds = l_i(T_0; t); \quad (2.2)$$

if  $N_i(t) > 0$ , then the integrated telegraph process is expressed as

$$\int_0^t \mathcal{T}_i(s) ds = \sum_{n=0}^{N_i(t)-1} l_{\varepsilon_i(\tau_n)}(T_n; \tau_n, \tau_{n+1}) + l_{\varepsilon_i(\tau_{N_i(t)})}(T_{N_i(t)}; \tau_{N_i(t)}, t). \quad (2.3)$$

Here

$$l_i(T; u, t) := \int_u^t c_i(T, s) ds, \quad i = 0, 1.$$

Notice that  $l_i(T; u, s) + l_i(T; s, t) \equiv l_i(T; u, t)$ ,  $i = 0, 1$ . Simplifying we write  $l_i(T; t)$  instead of  $l_i(T; 0, t)$ .

In the same manner we define the jump component. Let  $h_0 = h_0(T)$  and  $h_1 = h_1(T)$ ,  $T \geq 0$  be a pair of deterministic continuous (or, at least, boundary measurable) functions. Consider telegraph processes (2.1) based on  $h_i(T)$  instead of  $c_i = c_i(T, \cdot)$ ,  $i = 0, 1$ ,

$$\mathcal{T}_i(t; h_0, h_1) = \sum_{n=1}^{\infty} h_{\varepsilon_i(\tau_n)}(T_n) \mathbf{1}_{\{\tau_n < t \leq \tau_{n+1}\}}, \quad i = 0, 1.$$

An integrated jump process is defined in the form of compound Poisson process by the integral

$$\int_0^t \mathcal{T}_i(s; h_0, h_1) dN_i(s) = \sum_{n=1}^{N_i(t)} h_{\varepsilon_i(\tau_n)}(T_n), \quad i = 0, 1. \quad (2.4)$$

The amplitude of a jump depends on the time spent by the particle at the current state.

Finally, the generalised integrated jump-telegraph process is the sum of the integrated telegraph process defined by (2.2)-(2.3) and the jump component defined by (2.4):

$$X_i(t) = \int_0^t \mathcal{T}_i(s; c_0, c_1) ds + \int_0^t \mathcal{T}_i(s; h_0, h_1) dN_i(s), \quad t \geq 0, \quad i = 0, 1. \quad (2.5)$$

It describes the particle which moves, alternating the velocity regimes at random times  $\tau_n$ , starting from the origin at the velocity regime  $c_i$ . Each velocity reversal is accompanied by jumps of random amplitudes,  $X_i(t)$  is the current particle's position.

Conditioning on the first velocity reversal, notice that

$$\begin{aligned} X_0(t) &\stackrel{D}{=} l_0(T_0; t) \mathbf{1}_{\{\tau_1 > t\}} + [l_0(T_0; \tau_1) + h_0(\tau_1) + X_1(t - \tau_1)] \mathbf{1}_{\{\tau_1 < t\}}, \\ X_1(t) &\stackrel{D}{=} l_1(T_0; t) \mathbf{1}_{\{\tau_1 > t\}} + [l_1(T_0; \tau_1) + h_1(\tau_1) + X_0(t - \tau_1)] \mathbf{1}_{\{\tau_1 < t\}}. \end{aligned} \quad (2.6)$$

Here  $\stackrel{D}{=}$  denotes the equality in distribution. At each of two equalities the first term represents the movement without velocity reversal; the second one is the sum of three terms: the path till the first reversal, the jump value and the movement which is initiated after the first reversal.

The distribution of  $X(t)$ ,  $t > 0$  is separated into the singular and absolutely continuous parts.

The singular part of the distribution corresponds to the movement without any velocity reversals; let  $\mathbb{P}_i^{(0)}$ ,  $i = 0, 1$  be the respective conditional distribution, if the initial state  $i = \varepsilon_i(0)$  is fixed: for any Borel set  $A$  we set

$$\mathbb{P}_i^{(0)}(A) := \mathbb{P}(X_i(t) \in A, N_i(t) = 0), \quad i = 0, 1.$$

We denote the corresponding expectation by  $\mathbb{E}_i^{(0)}\{\cdot\}$ . On the space of (continuous) test-functions  $\varphi$  consider the linear functional (generalised function),  $\varphi \rightarrow \mathbb{E}_i^{(0)}\{\varphi(X(t))\}$ . It is easy to see that

$$\mathbb{E}_i^{(0)}\{\varphi(X(t))\} = \int_{-\infty}^{\infty} \varphi(y) \mathbb{P}_i^{(0)}(dy) = \bar{F}_i(t) \int_0^{\infty} \varphi(l_i(s; t)) f_{1-i}(s) ds =: \langle p_i(\cdot, t; 0), \varphi \rangle.$$

The generalised function

$$p_i(x, t; 0) = \bar{F}_i(t) \int_0^\infty \delta_{l_i(s; t)}(x) f_{1-i}(s) ds = \bar{F}_i(t) \int_0^\infty \delta_0(x - l_i(s; t)) f_{1-i}(s) ds \quad (2.7)$$

can be viewed as the distribution “density”. Here  $\delta_a(x)$  is the Dirac measure (of unit mass) at point  $a$ .

The absolutely continuous part of the distribution of  $X_i(t)$  is characterised by the densities

$$p_i(x, t; n) = \mathbb{P}\{X_i(t) \in dx, N_i(t) = n\}/dx, \quad i = 0, 1, \quad n \geq 1.$$

The sum

$$p_i(x, t) = \sum_{n=1}^{\infty} p_i(x, t; n)$$

corresponds to the absolutely continuous part of distribution of  $X_i(t)$ ,  $i = 0, 1$ .

Conditioning on the first velocity reversal, similarly to (2.6) we obtain the following equations,  $n \geq 1$ ,

$$\begin{aligned} p_0(x, t; n) &= \int_0^\infty f_1(\tau) d\tau \int_0^t p_1(x - l_0(\tau; s) - h_0(s), t - s; n - 1) f_0(s) ds, \\ p_1(x, t; n) &= \int_0^\infty f_0(\tau) d\tau \int_0^t p_0(x - l_1(\tau; s) - h_1(s), t - s; n - 1) f_1(s) ds \end{aligned} \quad (2.8)$$

(if  $n = 1$  the inner integrals are understood in the sense of the theory of generalised functions). Summing up in (2.8) we get the system of integral equations for (complete) distribution densities,

$$\begin{aligned} p_0(x, t) &= p_0(x, t; 0) + \int_0^\infty f_1(\tau) d\tau \int_0^t p_1(x - l_0(\tau; s) - h_0(s), t - s) f_0(s) ds, \\ p_1(x, t) &= p_1(x, t; 0) + \int_0^\infty f_0(\tau) d\tau \int_0^t p_0(x - l_1(\tau; s) - h_1(s), t - s) f_1(s) ds. \end{aligned} \quad (2.9)$$

Here  $p_0(x, t; 0)$  and  $p_1(x, t; 0)$  are defined by (2.7).

If  $c_0, c_1 \equiv \text{const}$ ,  $h_0, h_1 \equiv \text{const}$  equations (2.8) and (2.9) can be solved explicitly using the following notations,

$$\xi = \xi(x, t) := \frac{x - c_1 t}{c_0 - c_1} \quad \text{and} \quad t - \xi = \frac{c_0 t - x}{c_0 - c_1}.$$

Notice that  $0 < \xi(x, t) < t$ , if  $x \in (c_1 t, c_0 t)$  (say,  $c_0 > c_1$ ). Define the functions  $q_i(x, t; n)$ ,  $i = 0, 1$ : for  $c_1 t < x < c_0 t$ ,

$$\begin{aligned} q_0(x, t; 2n) &= \frac{\lambda_0^n \lambda_1^n}{(n-1)! n!} \xi^n (t - \xi)^{n-1} \\ q_1(x, t; 2n) &= \frac{\lambda_0^n \lambda_1^n}{(n-1)! n!} \xi^{n-1} (t - \xi)^n \end{aligned}, \quad n \geq 1, \quad (2.10)$$

and

$$\begin{aligned} q_0(x, t; 2n+1) &= \frac{\lambda_0^{n+1} \lambda_1^n}{(n!)^2} \xi^n (t - \xi)^n \\ q_1(x, t; 2n+1) &= \frac{\lambda_0^n \lambda_1^{n+1}}{(n!)^2} \xi^n (t - \xi)^n \end{aligned}, \quad n \geq 0. \quad (2.11)$$

Denote  $\theta(x, t) = \frac{1}{c_0 - c_1} e^{-\lambda_0 \xi - \lambda_1(t - \xi)} \mathbf{1}_{\{0 < \xi < t\}}$ .

Equations (2.8) have the following solution:

$$\begin{aligned} p_i(x, t; 0) &= e^{-\lambda_i t} (x - c_i t), \\ p_i(x, t; n) &= q_i(x - j_{in}, t; n) \theta(x - j_{in}, t), \quad n \geq 1, \quad i = 0, 1, \end{aligned} \quad (2.12)$$

where the displacements  $j_{in}$  are defined as the sum of alternating jumps,  $j_{in} = \sum_{k=1}^n h_{i_k}$ , where  $i_k = i$ , if  $k$  is odd, and  $i_k = 1 - i$ , if  $k$  is even.

Summing up we obtain the solution of (2.9):

$$\begin{aligned} p_i(x, t) &= e^{-\lambda_i t} \cdot \delta_0(x - c_i t) \\ &+ \frac{1}{c_0 - c_1} \left[ \lambda_i \theta(x - h_i, t) I_0 \left( 2 \frac{\sqrt{\lambda_0 \lambda_1 (c_0 t - x + h_i)(x - h_i - c_1 t)}}{c_0 - c_1} \right) \right. \\ &\left. + \sqrt{\lambda_0 \lambda_1} \theta(x, t) \left( \frac{x - c_1 t}{c_0 t - x} \right)^{\frac{1}{2} - i} I_1 \left( 2 \frac{\sqrt{\lambda_0 \lambda_1 (c_0 t - x)(x - c_1 t)}}{c_0 - c_1} \right) \right], \end{aligned} \quad (2.13)$$

where  $I_0(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{(n!)^2}$  and  $I_1(z) = I_0'(z)$  are the modified Bessel functions.

See the proof of (2.10)-(2.13) in Ratanov (2007).

### 3. Generalised jump-telegraph processes: moments

Using (2.9) the equations for the expectations can be derived also. Let  $\mu_i(t) := \mathbb{E}\{X_i(t)\}$  and  $\bar{l}_i(\cdot) := \mathbb{E}\{l_i(T; t)\} = \int_0^\infty f_{1-i}(\tau) l_i(\tau; t) d\tau$ ,  $t \geq 0$ . Equations (2.9) lead to

$$\mu_i(t) = \bar{F}_i(t) \bar{l}_i(t) + \int_0^t (\bar{l}_i(s) + h_i(s) + \mathbb{E}\{X_{1-i}(t - s)\}) f_i(s) ds, \quad i = 0, 1.$$

Therefore the expectations  $\mu_i$ ,  $i = 0, 1$  follow the equations of Volterra type:

$$\begin{aligned} \mu_0(t) &= a_0(t) + \int_0^t \mu_1(t - s) f_0(s) ds, \\ \mu_1(t) &= a_1(t) + \int_0^t \mu_0(t - s) f_1(s) ds, \end{aligned} \quad (3.1)$$

where

$$a_i(t) := \bar{F}_i(t) \bar{l}_i(t) + \int_0^t (\bar{l}_i(s) + h_i(s)) f_i(s) ds, \quad i = 0, 1.$$

Integrating by parts at the latter integral we have

$$\int_0^t \bar{l}_i(s) f_i(s) ds = -\bar{F}_i(t) \bar{l}_i(t) + \int_0^t \bar{c}_i(s) \bar{F}_i(s) ds,$$

which gives the following simplification for functions  $a_i$ :

$$a_i(t) = \int_0^t (\bar{F}_i(s) \bar{c}_i(s) + f_i(s) h_i(s)) ds. \quad (3.2)$$

Here we denote  $\bar{c}_i(s) = \mathbb{E}\{c_i(\cdot; s)\} = \int_0^\infty f_{1-i}(\tau) c_i(\tau; s) d\tau$ ,  $i = 0, 1$ .

Equations for variances  $\sigma_i(t) := \text{var}\{X_i(t)\} = \mathbb{E}\{(X_i(t) - \mu_i(t))^2\}$  can be derived similarly:

$$\begin{aligned}\sigma_0(t) &= b_0(t) + \int_0^t \sigma_1(t-s) f_0(s) ds, \\ \sigma_1(t) &= b_1(t) + \int_0^t \sigma_0(t-s) f_1(s) ds,\end{aligned}\tag{3.3}$$

where

$$b_i(t) := \bar{F}_i(t) (\bar{l}_i(t) - \mu_i(t))^2 + \int_0^t (\bar{l}_i(s) + h_i(s) + \mu_{1-i}(t-s) - \mu_i(t))^2 f_i(s) ds, \quad i = 0, 1.$$

Generalising (3.1)-(3.3), we have the following result.

**Theorem 3.1.** *Let  $g = g(x)$ ,  $-\infty < x < \infty$  be a locally bounded measurable function. Assume that*

$$\int_0^\infty f_{1-i}(\tau) |g(x + l_i(\tau; t))| d\tau < \infty, \quad i = 0, 1.\tag{3.4}$$

*Then the expectations*

$$u_0(x, t) = \mathbb{E}\{g(x + X_0(t))\}, \quad u_1(x, t) = \mathbb{E}\{g(x + X_1(t))\}$$

*exist, and they satisfy the system*

$$u_0(x, t) = G_0(x, t) + \int_0^\infty \int_0^t u_1(x + l_0(\tau; s) + h_0(s), t-s) f_1(\tau) f_0(s) d\tau ds,\tag{3.5}$$

$$u_1(x, t) = G_1(x, t) + \int_0^\infty \int_0^t u_0(x + l_1(\tau; s) + h_1(s), t-s) f_0(\tau) f_1(s) d\tau ds,\tag{3.6}$$

where  $G_i(x, t) = \bar{F}_i(t) \int_0^\infty f_{1-i}(\tau) g(x + l_i(\tau; t)) d\tau$ ,  $i = 0, 1$ .

*Proof.* Equations (3.5)-(3.6) follow by conditioning on the first velocity reversal, see (2.6).  $\square$

The equations for the moments  $\mu_i^{(N)}(t) := \mathbb{E}\{X_i(t)^N\}$ ,  $t \geq 0$ ,  $N \geq 0$  can be derived by using Theorem 3.1 with  $g(x) = x^N$ , see (3.5)-(3.6).

**Corollary 3.1.** *Let  $N = 0, 1, 2, \dots$*

*Functions  $\mu_0^{(k)}(t), \mu_1^{(k)}(t)$ ,  $t \geq 0, k = 0, 1, \dots, N$  satisfy the equations*

$$\begin{aligned}\mu_0^{(N)}(t) &= \bar{F}_0(t) \int_0^\infty f_1(\tau) l_0(\tau; t)^N d\tau + \sum_{k=0}^N \binom{N}{k} \int_0^t g_{0, N-k}(s) \mu_1^{(k)}(t-s) f_0(s) ds, \\ \mu_1^{(N)}(t) &= \bar{F}_1(t) \int_0^\infty f_0(\tau) l_1(\tau; t)^N d\tau + \sum_{k=0}^N \binom{N}{k} \int_0^t g_{1, N-k}(s) \mu_0^{(k)}(t-s) f_1(s) ds.\end{aligned}\tag{3.7}$$

Here  $g_{0,0} = g_{1,0} \equiv 1$  and

$$\begin{aligned}g_{0,m}(t) &= \int_0^\infty f_1(\tau) (l_0(\tau; t) + h_0(t))^m d\tau, \\ g_{1,m}(t) &= \int_0^\infty f_0(\tau) (l_1(\tau; t) + h_1(t))^m d\tau,\end{aligned} \quad m \geq 1.$$

In general, systems (3.1), (3.3) and (3.7) have the form of the recursive Volterra equations of the second kind:

$$\begin{aligned}\mu_0^{(N)}(t) &= a_0^{(N)}(t) + \int_0^t \mu_1^{(N)}(t-s) f_0(s) ds, \\ \mu_1^{(N)}(t) &= a_1^{(N)}(t) + \int_0^t \mu_0^{(N)}(t-s) f_1(s) ds,\end{aligned}\tag{3.8}$$

where  $a_i^{(N)}(t)$ ,  $i = 0, 1$  are generated by the preceding moments,  $\mu_{1-i}^{(k)}$ ,  $k = 0, \dots, N-1$ :

$$\begin{aligned}a_0^{(N)}(t) &:= \bar{F}_0(t) \int_0^\infty l_0(\tau; t)^N f_1(\tau) d\tau + \sum_{k=0}^{N-1} \binom{N}{k} \int_0^t g_{0, N-k}(s) \mu_1^{(k)}(t-s) f_0(s) ds, \\ a_1^{(N)}(t) &:= \bar{F}_1(t) \int_0^\infty l_1(\tau; t)^N f_0(\tau) d\tau + \sum_{k=0}^{N-1} \binom{N}{k} \int_0^t g_{1, N-k}(s) \mu_0^{(k)}(t-s) f_1(s) ds.\end{aligned}\tag{3.9}$$

Here  $N \geq 1$ .

System (3.8) possesses a unique solution, see e.g. Linz (1985). Under appropriate assumptions the solution can be found explicitly. Consider the following example. Let the distributions of interarrival times are exponential:

$$f_i(t) = \lambda_i \exp(-\lambda_i t), \quad t \geq 0, \quad i = 0, 1.$$

In this particular case system (3.8) is solved by

$$\boldsymbol{\mu}(t) = \boldsymbol{a}(t) + \int_0^t (I + \varphi(t-s)\Lambda) L \boldsymbol{a}(s) ds,\tag{3.10}$$

where  $\varphi(t) = (1 - e^{-2\lambda t})/(2\lambda)$ ,  $2\lambda := \lambda_0 + \lambda_1$ . Here we use the matrix notations  $\boldsymbol{\mu} = (\mu_0^{(N)}, \mu_1^{(N)})'$ ,  $\boldsymbol{a} = (a_0^{(N)}, a_1^{(N)})'$ ,

$$L = \begin{pmatrix} 0 & \lambda_0 \\ \lambda_1 & 0 \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}.$$

To check it, notice that system (3.8) is equivalent to ODE with zero initial condition:

$$\frac{d\boldsymbol{\mu}}{dt} = \Lambda \boldsymbol{\mu}(t) + \boldsymbol{\phi}(t), \quad \boldsymbol{\mu}(0) = 0,$$

where  $\boldsymbol{\phi} = \frac{d\boldsymbol{a}}{dt} + (L - \Lambda)\boldsymbol{a}$ . We get this equation by differentiating in (3.8) with subsequent integration by parts. Clearly, the unique solution is

$$\boldsymbol{\mu}(t) = \int_0^t e^{(t-s)\Lambda} \boldsymbol{\phi}(s) ds.\tag{3.11}$$

Integrating by parts in (3.11) we obtain

$$\boldsymbol{\mu}(t) = \boldsymbol{a}(t) + \int_0^t e^{(t-s)\Lambda} L \boldsymbol{a}(s) ds.$$

Now, the desired representation (3.10) follows from

$$\exp\{t\Lambda\} = I + \varphi(t)\Lambda = \frac{1}{2\lambda} \begin{pmatrix} \lambda_1 + \lambda_0 e^{-2\lambda t} & \lambda_0(1 - e^{-2\lambda t}) \\ \lambda_1(1 - e^{-2\lambda t}) & \lambda_0 + \lambda_1 e^{-2\lambda t} \end{pmatrix}.\tag{3.12}$$

#### 4. Martingales

Let  $X_0 = X_0(t)$  and  $X_1 = X_1(t)$  be (integrated) telegraph processes defined by (2.5) on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Let  $\mu_i(t) = \mathbb{E}\{X_i(t)\}$ ,  $i = 0, 1$  denotes the expectations, and coefficients  $a_i(t)$ ,  $i = 0, 1$  are defined by (3.2).

Notice that by (3.1)  $\mu_0 = \mu_1 \equiv 0$  if and only if  $a_0 = a_1 \equiv 0$ , which is equivalent to the set of identities, see (3.2),

$$\begin{aligned} \bar{F}_0(t)\bar{c}_0(t) + h_0(t)f_0(t) &\equiv 0 \\ \bar{F}_1(t)\bar{c}_1(t) + h_1(t)f_1(t) &\equiv 0, \end{aligned} \quad t \geq 0. \quad (4.1)$$

Let  $\mathcal{F}_t$ ,  $t \geq 0$  be the filtration, generated by  $\{(X_0(s), X_1(s)) \mid s \leq t\}$ .

**Theorem 4.1.** *The integrated jump-telegraph processes  $X_0$  and  $X_1$  defined by (2.5) are  $\mathcal{F}_t$ -martingales if and only if (4.1) holds.*

*Proof.* The proof can be done by computing the conditional expectation  $\mathbb{E}\{X_i(t_2) - X_i(t_1) \mid \mathcal{F}_{t_1}\}$  for  $0 \leq t_1 \leq t_2$ . Indeed,

$$\begin{aligned} \mathbb{E}\{X_i(t_2) - X_i(t_1) \mid \mathcal{F}_{t_1}\} &= \mathbb{E}\left\{\int_{t_1}^{t_2} \mathcal{T}_i(s; c_0, c_1)ds + \sum_{n=N_i(t_1)+1}^{N_i(t_2)} h_{\varepsilon_i(\tau_n)}(T_n) \mid \mathcal{F}_{t_1}\right\} \\ &= \mathbb{E}\left\{\int_0^{t_2-t_1} \mathcal{T}_{\varepsilon_i(t_1+s)}(t_1+s)ds + \sum_{n=1}^{N_i(t_2)-N_i(t_1)} h_{\varepsilon_i(\tau_{n+N_i(t_1)})}(T_{n+N_i(t_1)}) \mid \mathcal{F}_{t_1}\right\} \end{aligned}$$

According to the Markov property applied to the processes  $\varepsilon_i = \varepsilon_i(t)$ ,  $N_i = N_i(t)$  and  $\{\tau_k\}$  we have

$$\begin{aligned} \varepsilon_i(t_1+s) &\stackrel{D}{=} \tilde{\varepsilon}_{\varepsilon_i(t_1)}(s), & N_i(t_1+s) &\stackrel{D}{=} N_i(t_1) + \tilde{N}_{\varepsilon_i(t_1)}(s), & s \geq 0, \\ \tau_{n+N(t_1)} &\stackrel{D}{=} \tilde{\tau}_n, & T_{n+N(t_1)} &\stackrel{D}{=} \tilde{T}_n, & n \geq 1, \end{aligned}$$

where  $\tilde{\varepsilon}(s)$ ,  $\tilde{N}(s)$ ,  $\tilde{\tau}_n$  and  $\tilde{T}_n$  are copies of  $\varepsilon(s)$ ,  $N(s)$ ,  $\tau_n$  and  $T_n$  respectively, independent of  $\mathcal{F}_{t_1}$ . Therefore,

$$\mathbb{E}\{X_i(t_2) - X_i(t_1) \mid \mathcal{F}_{t_1}\} = \mathbb{E}\{\tilde{X}_{\varepsilon_i(t_1)}(t_2 - t_1)\}.$$

Here  $\tilde{X}_{\varepsilon_i(t_1)}$  denotes the integrated jump-telegraph process, which is initiated from the state  $\varepsilon_i(t_1)$ , and is based on  $\tilde{\varepsilon}(s)$ ,  $\tilde{N}(s)$ ,  $\tilde{\tau}_n$  and  $\tilde{T}_n$ . The latter expectation is equal to zero,  $\mathbb{E}\{\tilde{X}_{\varepsilon_i(t_1)}(t_2 - t_1)\} \equiv 0$ , if and only if (4.1) holds.  $\square$

**Remark 4.1.** *Notice that if (4.1) holds, then the direction of jump should be opposite to the (mean) velocity value.*

**Corollary 4.1.** *If the jump-telegraph processes  $X_0$  and  $X_1$  defined by (2.5) are martingales, then*

$$\frac{\bar{c}_i(t)}{h_i(t)} < 0 \quad \forall t \geq 0, \quad (4.2)$$

$$\int_0^\infty \frac{\bar{c}_i(s)}{h_i(s)} ds = \infty, \quad i = 0, 1. \quad (4.3)$$



Moreover,  $X_0$  and  $X_1$  are martingales, if and only if the distribution densities of interarrival times satisfy the following integral relations:

$$f_i(t) = -\frac{\bar{c}_i(t)}{h_i(t)} \exp \left\{ \int_0^t \frac{\bar{c}_i(s)}{h_i(s)} ds \right\}, \quad i = 0, 1. \quad (4.4)$$

*Proof.* Inequality (4.2) follows directly from (4.1). Identities (4.1) are equivalent to

$$\frac{\bar{c}_i(t)}{h_i(t)} = -\frac{f_i(t)}{\bar{F}_i(t)} \equiv (\ln \bar{F}_i(t))', \quad i = 0, 1. \quad (4.5)$$

Therefore

$$\bar{F}_i(t) = \exp \left\{ \int_0^t \frac{\bar{c}_i(s)}{h_i(s)} ds \right\}, \quad t \geq 0, \quad i = 0, 1.$$

The latter equality is equivalent to (4.4).

Notice that by definition  $\lim_{t \rightarrow +\infty} \bar{F}_i(t) = 0$ . Hence, condition (4.3) is fulfilled.  $\square$

In this framework various particular cases of the martingale distributions and the corresponding distributions of interarrival times can be presented by applying Corollary 4.1. Consider the following examples.

*Exponential distribution.* Assume that functions  $\bar{c}_i(t)$  and  $h_i(t)$  are proportional:

$$\frac{\bar{c}_i(t)}{h_i(t)} \equiv -\lambda_i, \quad \lambda_i > 0, \quad i = 0, 1. \quad (4.6)$$

Relations (4.4) mean that the integrated jump-telegraph process is the martingale if the distributions of interarrival times are exponential:  $f_i(t) = \lambda_i \exp(-\lambda_i t)$ ,  $t > 0$ ,  $i = 0, 1$ .

Identities (4.6) can be written in detail as follows. The (observable) parameters of the model, i. e. the regimes of velocities  $c_0, c_1$  and the regimes of jumps  $h_0, h_1$ , satisfy the equations

$$\lambda_1 \int_0^\infty e^{-\lambda_1 \tau} c_0(\tau, t) d\tau = -\lambda_0 h_0(t), \quad \lambda_0 \int_0^\infty e^{-\lambda_0 \tau} c_1(\tau, t) d\tau = -\lambda_1 h_1(t)$$

with some positive constants  $\lambda_0$  and  $\lambda_1$ . These equations help to compute the switching intensities  $\lambda_0$  and  $\lambda_1$  by using the (observable) proportion between velocity and jump values. On the other hand, if mean velocity regimes are given,  $\bar{c}_0$  and  $\bar{c}_1$ , from these equations we can conclude that small jumps occur with high frequency, and big jumps are rare. The direction of jump should be opposite to the velocity sign, see also Remark 4.1.

**Proposition 4.1.** *In the framework of (2.5) we assume that the Markov flow of switching times  $\mathfrak{T} = \{\tau_k\}_{k=0}^\infty$  has interarrival intervals  $\tau_k - \tau_{k-1}$ ,  $k \geq 1$  which are exponentially distributed with alternated constant intensities  $\mu_0, \mu_1 > 0$ . Let the velocity regimes  $c_i = c_i(t)$  and jump amplitudes  $h_i = h_i(t)$  are given, and they are proportional as in (4.6),  $c_i(t)/h_i(t) = -\lambda_i$ ,  $i = 0, 1$ .*

*The martingale measure for  $(X_0, X_1)$  exists and it is unique.*

*Proof.* According to the Girsanov Theorem, see Ratanov (2007), we apply Radon-Nikodym derivative of the form

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}_t\{X^*\} = \exp \left\{ \int_0^t \mathcal{T}_i(s; c_0^*, c_1^*) ds \right\} \kappa_i^*(t), \quad (4.7)$$

where  $\kappa_i^*(t) = \prod_{k=1}^{N_i(t)} (1 + h_{\varepsilon_i(\tau_{k-1})}^*)$  is produced by the jump process with constant jump amplitudes  $h_i^* = -c_i^*/\mu_i$ , and  $\int_0^t \mathcal{T}_i(s; c_0^*, c_1^*) ds$  is the integrated telegraph process with constant velocities  $c_i^* = \mu_i - \lambda_i$ . Under the new measure  $\mathbb{Q}$  the underlying Markov flow takes the intensities  $\lambda_i$ ,  $i = 0, 1$  (see Theorem 2 and Theorem 3 by Ratanov (2007)). Therefore, process  $X_i(t)$  becomes the martingale.  $\square$

*Erlang distribution.* Telegraph processes with Erlang-distributed interarrival times have been studied by Perry et al (1999) and Di Crescenzo (2001). In our setting, it is easy to see that the martingale distribution can be obtained by means of alternated Erlang distribution for interarrival times,  $f_i(t) = \frac{\lambda_i^{n_i} t^{n_i-1}}{(n_i-1)!} e^{-\lambda_i t} \mathbf{1}_{\{t>0\}}$ ,  $\bar{F}_i(t) = \sum_{k=0}^{n_i-1} (\lambda_i t)^k / k!$ ,  $\lambda_i > 0$ ,  $n_i \geq 1$ ,  $i = 0, 1$ , if the velocities and jumps follow the proportion, see (4.5),

$$\bar{c}_i(t)/h_i(t) = -\frac{\lambda_i^{n_i} t^{n_i-1} / (n_i-1)!}{\sum_{k=0}^{n_i-1} (\lambda_i t)^k / k!}. \quad (4.8)$$

One can get the martingale measure by changing the intensities of the underlying Poisson process (see Proposition 4.1).

More precisely, let  $(\Omega, \mathcal{F}, \mathbb{P})$  be given probability space. Consider the Poisson flow  $\mathfrak{T} = \{\tau_k\}_{k=0}^\infty$  with constant switching intensities  $\mu_0, \mu_1 > 0$ . Let  $\mathcal{G}_t$ ,  $t \geq 0$  be a filtration based on this Poisson flow.

We interpret the governing Erlang flow  $\mathfrak{T}^{(n)}(t)$  as thinned Poisson: the system accepts each  $n$ -th arrived signal. Let  $\mathcal{F}_t$  be the filtration generated by  $\{\mathfrak{T}^{(n)}(s) : s \leq t\}$ . Clearly,  $\mathcal{F}_t \subset \mathcal{G}_t$ ,  $\forall t \geq 0$ .

All filtrations here are assumed to satisfy the usual hypotheses, see Protter (2005).

Changing the measure by means of the Radon-Nikodym derivative defined by (4.7) we pass from intensities  $\mu_0, \mu_1$  to intensities  $\lambda_0, \lambda_1$  (defined by (4.8)) for the underlying Poisson process. Therefore, under the new measure the telegraph process with jumps,  $X = X(t)$ ,  $t \geq 0$  is  $\mathcal{G}_t$ -martingale. Then  $X = X(t)$ ,  $t \geq 0$  is again martingale, for the filtration  $\mathcal{F}_t$ , see Theorem 2.2, Föllmer and Protter (2011).

Another particular possibilities are the following.

1. *Weibull distribution.* Assuming that

$$\bar{c}_i(t)/h_i(t) = -\lambda_i t^{\alpha_i}, \quad \alpha_i > -1, \lambda_i > 0, i = 0, 1,$$

we have  $f_i(t) = \lambda_i t^{\alpha_i} \exp \left\{ -\frac{\lambda_i}{\alpha_i+1} t^{\alpha_i+1} \right\} \mathbf{1}_{t>0}$ .

2. *Pareto distribution.* Let  $0 < \lambda_0, \lambda_1 < 2$ . For  $b_0, b_1 > 0$  assume that

$$\bar{c}_i(t)/h_i(t) = -\frac{\lambda_i}{t} \cdot \mathbf{1}_{\{t>b_i\}}, \quad i = 0, 1.$$

Hence, the martingale distribution is determined by a Pareto distribution for interarrival times, i. e.  $f_i(t) = \lambda_i b_i^{\lambda_i} t^{-1-\lambda_i} \mathbf{1}_{\{t>b_i\}}$ ,  $i = 0, 1$ .

This distribution is in the domain of normal attraction of some  $\lambda_i$ -stable distribution, see Feller (1971).

3. *Logistic distribution.* Let interarrival times  $T_n$ ,  $n \in \mathbb{Z}$  have (alternated) logistic distributions with the density  $f_i(t) = \frac{2\lambda_i e^{-\lambda_i t}}{(1+e^{-\lambda_i t})^2} \mathbf{1}_{\{t \geq 0\}}$ , see Di Crescenzo and Martinucci (2010). This produces the martingale distribution, if

$$\bar{c}_i(t)/h_i(t) = -\frac{\lambda_i e^{-\lambda_i t}}{1 + e^{-\lambda_i t}}, \quad t \geq 0.$$

4. *Cauchy distribution.* The distribution  $f_i$  takes the form of Cauchy, such that  $f_i(t) = \frac{2a_i/\pi}{a_i^2+t^2} \mathbf{1}_{\{t \geq 0\}}$ , if

$$\bar{c}_i(t)/h_i(t) = -\frac{a_i}{(a_i^2+t^2)(\frac{\pi}{2} - \arctan(t/a_i))}, \quad t \geq 0.$$

5. *Uniform distribution.* Let

$$\bar{c}_i(t)/h_i(t) = -\frac{1}{A_i-t} \mathbf{1}_{0 < t < A_i}.$$

Then, in this blow-up case, we have the uniform distribution,  $f_i(t) = \frac{1}{A_i} \mathbf{1}_{0 < t < A_i}$ .

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